# **Self-organization of Fractal Networks**

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## 1. Introduction



complex systems



#### complex networks



How complex are real-world networks?

• Large fluctuation of the number of edges from a node (degree)

 $P(k) \propto k^{-\gamma} \quad (k \gg 1)$ 

Scale-free property



Hub nodes exist



• Community structures

Degree correlations





assortative mixing

disassortative mixing





### Examples of small-world networks

Network	N	$\langle l \rangle$
Internet	10,697	3.31
co-authorship	1,520,251	4.6
word co-occurrence	478,773	2.63
airline	3,880	4.37



Image of airline network http://www.sita.aero/

## Origin of the small-world property

[D.J. Watts and S.H. Strogatz, Nature 393, 440 (1998).]



#### short-cut edges



#### Fractal networks

$$N_{\rm B}(l_{\rm B}) \propto l_{\rm B}^{-D}$$

$$\begin{pmatrix} N_B(l_B) : \text{number of subgraphs} \\ \text{of diameter } l_B \\ l_B : \text{subgraph diameter} \end{pmatrix}$$

#### Examples of fractal networks



In addition, actor networks, protein interaction networks, power grid networks, subway networks, software networks, etc. are fractal.

### Origin of fractality

???

It has not yet been clarified how complex networks become fractal.

"Mechanism of fractal network formation" ( Open question

More fundamental question:

Why are complex networks classified into two classes, small-world and fractal networks?

### Fractal objects in nature



## Self-organized criticality (SOC)

[P. Bak et al., Phys. Rev. Lett. 59, 381 (1987).]

Property of dynamical systems that evolve spontaneously to a critical point and fluctuate around it.



- Intermittent avalanches
- Power-law distribution of avalanche interval
- Power-law distribution of avalanche size
- Power-law distribution of cluster size

Emergence of fractal structures (sand-pile, river basin, coast line, etc.)





#### Previous studies of SOC dynamics on networks

- E. Bonabeau, J. Phys. Soc. Jpn. **64**, 327 (1995).
- Y. Moreno *et al.*, Physica A **274**, 400 (1999).
- K.-I. Goh *et al.*, PRL **91**, 148701 (2003).
- N. Masuda *et al.*, PRE **72**, 066106 (2005).
- F. Caruso *et al.*, Eur. Phys. J. B **50**, 243 (2006).
- G. L. Pellegrini *et al.*, PRE **76**, 016107 (2007).
- B. Luque *et al.*, PRL **101**, 158702 (2008).
- A. Watanabe and K. Y, PRE **89**, 052806 (2014).

Internal degrees of freedom evolve on a *fixed* network structure.

To form fractal networks by SOC dynamics,



#### Previous studies of interplay between structures and internal variables

- K. Christensen *et al.*, PRL **81**, 2380 (1998).
- F. Slanina and M. Kotrla, PRL 83, 5587 (1999).
- M. Paczuski and D. Hughes, Physica A **342**, 158 (2004).
- D. Garlaschelli *et al.*, Nat. Phys. **3**, 813 (2007).
- C. Guill and B. Drossel, J. Theor. Biol. **251**, 108 (2008).
- M. Rybarsch and S. Bornholdt, PLOS ONE 9, e93090 (2014).
- T. Shimada, Sci. Rep. 4, 4082 (2014).



Network model of solar flare dynamics [M. Paczuski et al., Physica A **342**, 158 (2004).]

#### ✓ non-intermittent dynamics

- ✓ parameter tuning
- ✓ non-fractal
- $\checkmark$  fixed *N* or *M*

#### Our purpose

Constructing an SOC model to explain the spontaneous formation of fractal networks through the interplay between network structures and internal degrees of freedom.

not **SOC**.

## 2. Model

SOC dynamics of network

 $\implies$  Network size (N) fluctuates around a steady size.

Size fluctuation Size fluctuat

### Network collapses in the real world



### Functional networks, flow, and loads

Functions of networks are realized by flow.

Functional network	flow	loads
Internet	packets	packets
Traffic network	cars	cars
Power grid	current	current
Trading network	money	money (debt)

Flow also plays a role of load!











#### Previous studies on cascading overload failures

- M.L. Sachtjen, et al., Phys. Rev. E 61, 4877 (2000).
- A.E. Motter and Y.-C. Lai, Phys. Rev. E 66, 065102 (2002).
- Y. Moreno, et al., Europhys. Lett. **58**, 630 (2002).
- A.E. Motter, Phys. Rev. Lett., **93**, 098701 (2004).
- P. Crucitti, et al., Phys. Rev. E **69**, 045104 (2004).
- I. Dobson, et al., Prob. Eng. Info. Sci. **19**, 15 (2005).
- I. Simonsen, et al., Phys. Rev. Lett. **100**, 218701 (2008).
- R.-R. Liu, et al., Phys. Rev. E, **85**, 026110 (2012).
- L. Daqin, et al., Sci. Rep., **4**, 5381 (2014).

Overload failures  $\triangleleft$  <u>average load</u> > node capacity

#### "Scale-free networks are fragile against cascading overload failures!"

#### However,



#### **Overload failure**



## Random walker model of fluctuating loads

[V. Kishore et al., Phys. Rev. Lett. 106, 188701 (2011).]

### Temporally fluctuating loads

Random walkers

Number of walkers  $w_i(t)$  on node i at time t

Load of node *i* at time *t* 

Condition for overload failure  $w_i(t) > q_i$  (capacity of node i)

Natural setting of the node capacity

 $q_i = \langle w_i \rangle + m\sigma_i \qquad \begin{pmatrix} \langle w_i \rangle : \text{Average load on node } i \\ \sigma_i : \text{Fluctuation of load (std. dev.) on node } i \\ m : \text{Constant (node tolerance parameter)} \end{pmatrix}$ 

Overload probability  $F_W(k)$  of a node with degree k under the total load  $W_0$   $F_{W_0}(k) = I_{k/2M_0}([q_k(W_0)] + 1, W_0 - [q_k(W_0)])$  $\begin{bmatrix} I_x(a, b) : \text{Regularized incomplete beta function} \\ M_0 : \text{Number of edges} \end{bmatrix}$ 

# Model of cascading overload failures

[S. Mizutaka and K.Y., Phys. Rev. E **92**, 012814 (2015).]

- 1. Remove nodes from an initial network of N nodes, M edges, and the total load W = aM with the overload probability  $F_W(k) = I_{k/2M}([q_k] + 1, W - [q_k])$ . (First cascade step)
- 2. At step  $\tau$  of the cascade process, renew the total load by

 $W_{\tau} = \left(\frac{M_{\tau}}{M}\right)^{r} W$ ,  $\begin{pmatrix} M_{\tau} : \text{Number of edges in the network at step } \tau \\ r : \text{Load reduction parameter} \end{pmatrix}$ 

In reality, ✓ To avoid a large-scale chain bankruptcy in a company trading network,

Insurances, loans Safety-net systems Injection of public money



- ✓ To avoid a large-scale blackout in a power grid,
   Usage restriction of electricity
- 3. Calculate the overload probability under the new total load  $W_{\tau}$ , and remove nodes with this probability.
- 4. Repeat 2 and 3 until no nodes are removed.



### Model of network evolution

- 1. Prepare an initial small network of  $N_{ini}$  nodes and  $M_{ini}$  edges Total load :  $W_{ini} = aM_{ini}$ Node capacity :  $q_k (W_{ini}) = \langle w \rangle_k + m\sigma_k$
- 2. At each time step t, add a node with  $\mu$  edges, and connect it to existing nodes randomly.

Total load :  $W_0(t) = aM_0(t)$ Capacity of the new node :  $q_\mu [W_0(t)]$ 



3. Perform cascading overload failures until no nodes are removed with the overload probability  $F_{W_{\tau}(t)}(k)$  and



### Model of network evolution

- 1. Prepare an initial small network of  $N_{ini}$  nodes and  $M_{ini}$  edges Total load :  $W_{ini} = aM_{ini}$ Node capacity :  $q_k(W_{ini}) = \langle w \rangle_k + m$
- 2. At each time step t, add a node with  $\mu$  edges connect it to existing nodes randomly.

Total load :  $W_0(t) = aM_0(t)$ Capacity of the new node :  $q_\mu[W_0(t)]$ 



3. Perform cascading overload failures until no nodes are removed with the overload probability  $F_{W_{\tau}(t)}(k)$  and



## 3. Results



 $q_k(W_0) = \langle w \rangle_k + m\sigma_k$  $W_0(t) = aM_0(t)$ 



N(t) approaches to a critical size, then collapses.







#### Fractal and small-world structures

### Crossover from fractal to small-world



## Conclusions

We have proposed an SOC model of complex networks by combining network growth and cascading overload failures to explain spontaneous formation of fractal network structures.

Networks after large-scale cascades exhibit the fractal property.

#### Further research

- ✓ SOC model for scale-free fractal networks
- $\checkmark$  Simplification of the model
- $\checkmark$  Identification of the universality class